Dissipative state formulations and numerical simulation of a porous medium for boundary absorbing control of aeroacoustic waves

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Abstract: The problem under consideration relates to a model of porous wall devoted to aircraft motors noise reduction. For such a medium, the parameters of the propagation equation depend on the frequency, so the corresponding time-model involves non rational convolution operators. Consequently, the impedance of the wall is a non rational function of the frequency. On the basis of complex analysis and causality properties, we introduce infinite dimensional state formulations of these operators. The coupling of the so-obtained model with a standard aeroacoustic one then leads to a time-local system whose analysis is simplified thanks to the existence of an energy functional in the sense of which the global dissipativity is insured. Some numerical results are given to illustrate the theoretical results.

Keywords: porous medium, aeroacoustic waves, boundary control, diffusive representation, state representation, non rational operators, energy functional, energy balance.

1. INTRODUCTION

Aircraft motors noise reduction is currently an important challenge for aerospace industry. In the case of hot zones such as exhaust nozzles, a specific porous wall was proposed in Gasser [2003] for absorption of a wide part of the energy of incident acoustic waves. The frequency model of such a material which describes the wave propagation inside the porous medium, is given on $(\omega, z, x) \in \mathbb{R}\times]0,1[\times]0,X[$ by:

$$\begin{cases} e(x)\, i\omega\, \rho_{\text{eff}}\,(i\omega)\ \hat{u} + \partial_z \hat{P} = 0 \\ e(x)\, i\omega\, \chi_{\text{eff}}\,(i\omega)\ \hat{P} + \partial_z \hat{u} = 0, \end{cases} \tag{1}$$

where $\widehat{u}$ and $\widehat{P}$ designate the Fourier transforms of the velocity and the pressure in the porous medium, $e(x)$ denotes the thickness of the porous wall and the parameters $\rho_{\text{eff}}\,(i\omega)$ and $\chi_{\text{eff}}\,(i\omega)$ are respectively the so-called effective density of Pride *et al.* (Pride [1993]) and the effective compressibility of Lafarge (Lafarge [1993]). These parameters are expressed (Gasser [2003]):

$$\begin{cases} \rho_{\text{eff}}\,(i\omega) = \rho\,(1 + a\,\frac{(1+b\,i\omega)^{\frac{1}{2}}}{i\omega}) \\ \chi_{\text{eff}}\,(i\omega) = \chi\,(1 - c\,\frac{i\omega}{i\omega + a'(1+b' i\omega)^{\frac{1}{2}}}), \end{cases} \tag{2}$$

with:

$$\rho = \rho_0\,\alpha_\infty,\ \chi = \frac{1}{P_0},\ a = \frac{8\mu}{\rho_0\Lambda^2},\ a' = \frac{8\mu}{\rho_0\Lambda'^2},$$

$$b = \frac{1}{2a},\ b' = \frac{1}{2a'},\ 0 < c = \frac{\gamma - 1}{\gamma} < 1,$$

where the physical parameters ρ_0, P_0, μ, γ, α_∞, Λ, Λ' are respectively the density and pressure at rest, the dynamic viscosity, the specific heat ratio, the tortuosity, the high frequency characteristic length of the viscous incompressible problem and the high frequency characteristic length of the thermal problem.

The model (1) is completed by the following limit conditions:

$$\begin{cases} u_{|z=1} = 0 \quad \text{(wave reflection at } z = 1) \\ P_{|z=0} = w. \end{cases} \tag{3}$$

When coupling this system with the fluid medium, two connection conditions at the interface Γ are necessary:

$$P^{\text{fluid}}_{|\Gamma} = P_{|z=0}$$
$$\text{and } u^{\text{fluid}}_{|\Gamma} \cdot n = \phi\, u_{|z=0},$$

where n is the outgoing unit normal on Γ and $\phi > 0$ is the porosity coefficient of the material (Gasser [2003]). Then, w and

$$y := u_{|z=0} \tag{4}$$

can be interpreted respectively as the input and the output of (1), the correspondence $w \mapsto y$ defining the impedance operator of the porous wall.

Parameters ρ_{eff} and χ_{eff} depend on the frequency ω, so the time model obtained by inverse Fourier transform of (1), is written:

$$\begin{cases} e(x)\ (\partial_t \circ \rho_{\text{eff}}\,(\partial_t))\ u + \partial_z P = 0 \\ e(x)\ (\partial_t \circ \chi_{\text{eff}}\,(\partial_t))\ P + \partial_z u = 0, \end{cases} \tag{5}$$

where $\rho_{\text{eff}}\,(\partial_t)$ and $\chi_{\text{eff}}\,(\partial_t)$ are the (causal) convolution operators associated to the symbols ρ_{eff} and χ_{eff} respectively. Model (5) is not time-local because of the presence of convolution operators $\rho_{\text{eff}}\,(\partial_t)$ and $\chi_{\text{eff}}\,(\partial_t)$.

We propose in section 2 a new formulation of such operators by means of a suitable diffusive input-output differential system. This formulation allows us to obtain, in section 3, a time-local augmented input-output model equivalent to (1,3,4). We show that in the sense of an explicitly known functional, this model is consistent from the energy dissipation point of view. Then in section 4, the dissipativity of the whole problem is easily deduced from a suitable global energy functional. In section 5, we compute the analytic expression of the porous wall impedance from which we build a reduced diffusive state formulation of the associated operator. Finally, some numerical results and simulations are presented in section 6 .

2. DIFFUSIVE REALIZATION OF CAUSAL CONVOLUTION OPERATORS

In this section, we present a particular case of a methodology introduced and developed in Montseny [2005] in a general framework.

We consider a causal operator defined, on any continuous function $w : \mathbb{R}^+ \to \mathbb{R}$, by

$$w \mapsto \int_0^t h(t-s)\, w(s)\, ds. \tag{6}$$

We denote H the Laplace transform of h and $H(\partial_t)$ the convolution operator defined by (6).

Let $w^t(s) = \mathbf{1}_{]-\infty,t]}(s)\, w(s)$ and $w_t(s) = w^t(t-s)$. From causality of $H(\partial_t)$, we deduce:

$$\left(H(\partial_t)(w - w^t)\right)(t) = 0 \text{ for all } t;$$

then, we have for any continuous function w:

$$(H(\partial_t)w)(t) = \left[\mathcal{L}^{-1}\left(H\,\mathcal{L}w\right)\right](t) = \left[\mathcal{L}^{-1}\left(H\,\mathcal{L}w^t\right)\right](t). \tag{7}$$

We define:

$$\Psi_w(t,p) := \mathrm{e}^{pt}\left(\mathcal{L}w^t\right)(p) = \left(\mathcal{L}w_t\right)(-p); \tag{8}$$

by computing $\partial_t \mathcal{L} w_t$, Laplace inversion and use of (7), it can be shown:

Lemma 1. 1. The function Ψ_w is solution of the differential equation:

$$\partial_t \Psi(t,p) = p\,\Psi(t,p) + w, \quad t > 0, \quad \Psi(0,p) = 0. \tag{9}$$

2. For any $b \geqslant 0$,

$$\left(H(\partial_t)w\right)(t) = \frac{1}{2i\pi}\int_{b-i\infty}^{b+i\infty} H(p)\,\Psi_w(t,p)\,dp. \tag{10}$$

We denote Ω the holomorphic domain of H. Let γ a closed [1] simple arc in $\mathbb{C}^-$; we denote Ω_γ^+ the exterior domain defined by γ, and Ω_γ^- the complementary of $\overline{\Omega_\gamma^+}$. By use of standard techniques (Cauchy theorem, Jordan lemma (Lavrentiev [1977])), it can be shown:

Lemma 2. For $\gamma \subset \Omega$ such that H is holomorphic in Ω_γ^+, if $H(p) \to 0$ when $p \to \infty$ in Ω_γ^+, then:

$$\left(H(\partial_t)w\right)(t) = \frac{1}{2i\pi}\int_{\tilde\gamma} H(p)\,\Psi_w(t,p)\,dp, \tag{11}$$

where $\tilde\gamma$ is any closed simple arc in Ω_γ^+ such that $\gamma \subset \Omega_{\tilde\gamma}^-$.

We now suppose that γ, $\tilde\gamma$ are defined by functions of $W^{1,\infty}_{\mathrm{loc}}(\mathbb{R};\mathbb{C})$, also denoted γ, $\tilde\gamma$. Under hypothesis of lemma 2, we have (Montseny [2005]):

[1] Possibly at infinity

Theorem 3. If the possible singularities of H on γ are simple poles or branching points such that $|H \circ \gamma|$ is locally integrable, then:

1. with $\tilde\nu = \frac{\tilde\gamma'}{2\mathrm{i}\pi}\, H \circ \tilde\gamma$ and $\tilde\psi(t,.) = \Psi_w(t,.) \circ \tilde\gamma$:

$$\left(H(\partial_t)w\right)(t) = \int_{\mathbb{R}} \tilde\nu(\xi)\,\tilde\psi(t,\xi)\,d\xi; \tag{12}$$

2. with $\tilde\gamma_n \to \gamma$ in $W^{1,\infty}_{\mathrm{loc}}$ and $\nu = \frac{\tilde\gamma'}{2\mathrm{i}\pi} \lim H \circ \tilde\gamma_n$ in the sense of measures:

$$\left(H(\partial_t)w\right)(t) = \langle \nu, \psi(t,.)\rangle, \tag{13}$$

where $\psi(t,\xi)$ is solution of the evolution problem on $(t,\xi) \in \mathbb{R}^{*+} \times \mathbb{R}$:

$$\partial_t\psi(t,\xi) = \gamma(\xi)\,\psi(t,\xi) + w(t),\ \psi(0,\xi) = 0. \tag{14}$$

Remark 4. In the sequel, we will indifferently denote $\langle \nu, \psi\rangle$ or $\int \nu\,\psi\,d\xi$ the duality scalar product between a continuous function ψ and a measure ν.

Finally, in the particular case $\gamma(\xi) = -|\xi|$, we deduce from symmetry of the problem that there exists a measure μ such that

$$\int_{-\infty}^{+\infty} \nu\,\psi\,d\xi = \int_0^{+\infty} \mu\,\psi\,d\xi.$$

The state equation (14) is infinitedimensional. To get numerical approximations, we consider a discretization $(\xi_l)_{l=1:L}$ of the variable ξ and approximations μ_L of the γ-symbol μ using atomic measures and of the form:

$$\mu_L = \sum_{l=1}^{L} \mu_L^l \delta_{\xi_l}.$$

Let us denote $\mathcal{M}_L$ the space of atomic measures on the mesh $\{\xi_l\}_{l=1:L}$. If $\cup_L \mathcal{M}_L$ is dense in the space of measures, we have:

$$\langle \mu_L, \psi\rangle \underset{L\to+\infty}{\longrightarrow} \langle \mu, \psi\rangle \ \ \forall\psi.$$

So we get the finitedimensional approximate state formulation of $H(\partial_t)$:

$$\begin{cases} \partial_t\psi(t,\xi_l) = \gamma(\xi_l)\,\psi(t,\xi_l) + u(t),\ l = 1:L \\ \psi(0,\xi_l) = 0,\ l = 1:L \\ \left(H(\partial_t)u\right)(t) \simeq \sum\limits_{l=1}^{L} \mu_L^l\,\psi(t,\xi_l). \end{cases} \tag{15}$$

More details can be found in Montseny [2005].

3. STATE FORMULATION OF THE POROUS MEDIUM MODEL

Consider the convolution operators $H_1(\partial_t)$ and $H_2(\partial_t)$ with respective symbols

$$H_1(p) = 1/(p\,\rho_{\mathrm{eff}}(p)) \text{ and } H_2(p) = 1/(p\,\chi_{\mathrm{eff}}(p)).$$

These functions are decreasing at infinity and analytic in $\mathbb{C}\smallsetminus\mathbb{R}^-$. So, by considering the contour γ defined by $\gamma(\xi) = -|\xi|$, we can show that operators $H_1(\partial_t)$ and $H_2(\partial_t)$ satisfy the hypothesis of theorem 3. After computations as described in section 2, we obtain the following measures μ_1 and μ_2 respectively associated to $H_1(\partial_t)$ and $H_2(\partial_t)$:

$$\mu_1(\xi) = \frac{a}{\pi\,\rho}\frac{\sqrt{b\,\xi-1}}{\xi^2+\frac{a\,\xi}{2}-a^2}\mathbf{1}_{\xi>2a} + k_1\,\delta_{\xi_1}(\xi),$$

$$\mu_2(\xi) = \frac{a'\,c}{\pi\,\chi}\frac{\sqrt{b'\,\xi-1}}{\xi^2\,(1-c)^2+\frac{a'}{2}\xi-a'^2}\mathbf{1}_{\xi>2a'} + \frac{1}{\chi}\delta_0(\xi) + k_2\,\delta_{\xi_2}(\xi),$$

where:

$$\xi_1 = \tfrac{a(\sqrt{17}-1)}{4} > 0,\ \xi_2 = \tfrac{a'(\sqrt{1+16(1-c)^2}-1)}{4(1-c)^2} > 0,$$

$$k_1 = \tfrac{\sqrt{17}-1}{\rho\sqrt{17}} > 0,\ \text{and } k_2 = \tfrac{c\left(\sqrt{1+16(1-c)^2}-1\right)}{\chi(1-c)\sqrt{1+16(1-c)^2}} > 0.$$

Let us express the system (1) under the form:

$$\begin{cases} u = -H_1(\partial_t)\,\partial_z P/e \\ P = -H_2(\partial_t)\,\partial_z u/e. \end{cases}$$

From results of section 2, we get the following diffusive formulations of operators $\partial_z P \mapsto u$ and $\partial_z u \mapsto P$:

$$\begin{cases} \partial_t\psi_1 = -\xi\,\psi_1 - \dfrac{1}{e}\,\partial_z P \\ u := \langle \mu_1, \psi_1\rangle\,, \end{cases} \text{and} \begin{cases} \partial_t\psi_2 = -\xi\,\psi_2 - \dfrac{1}{e}\,\partial_z u \\ P := \langle \mu_2, \psi_2\rangle\,. \end{cases}$$

Then we get the following augmented model, defined on $(t,z,x,\xi) \in \mathbb{R}^{*+}\times]0,1[\times\Gamma\times\mathbb{R}^+$ and input-output equivalent to (1,3,4):

$$\begin{cases} \partial_t\psi_1 = -\xi\,\psi_1 - \dfrac{1}{e}\ \langle\mu_2, \partial_z\psi_2\rangle \\ \partial_t\psi_2 = -\xi\,\psi_2 - \dfrac{1}{e}\ \langle\mu_1, \partial_z\psi_1\rangle \\ \langle\mu_1, \psi_1(t,1,x,.)\rangle = 0 \\ \langle\mu_2, \psi_2(t,0,x,.)\rangle = w(t,x) \\ y(t,x) = \langle\mu_1, \psi_1(t,0,x,.)\rangle\,. \end{cases} \tag{16}$$

As μ_1 and μ_2 are positive, the functional

$$(\psi_1,\psi_2)\mapsto\|(\psi_1,\psi_2)\|_{\mu,x} = [\iint(\mu_1\,|\psi_1|^2 + \mu_2\,|\psi_2|^2)d\xi dz]^{\frac{1}{2}}$$

is a semi-norm. We have:

Proposition 5. The functional $E_{\psi_x} = \frac{1}{2}\|(\psi_1,\psi_2)\|^2_{\mu,x}$ verifies on any solution (ψ_1,ψ_2) of (16) :

$$\frac{dE_{\psi_x}}{dt} = -\int_0^1\int_0^\infty \xi\,\mu_1\,|\psi_1|^2\,d\xi\,dz - \int_0^1\int_0^\infty \xi\,\mu_2\,|\psi_2|^2\,d\xi\,dz + \frac{1}{e}\,w\,y. \tag{17}$$

Proof.

$$\begin{aligned} dE_{\psi_x}/dt &= -\sum_i\int_0^1\int_0^\infty \xi\,\mu_i\,|\psi_i|^2\,d\xi\,dz \\ &-\frac{1}{e}\int_0^1 P\,\partial_z u\,dz - \frac{1}{e}\int_0^1 \partial_z P\,u\,dz \\ &= -\sum_i\int_0^1\int_0^\infty \xi\,\mu_i\,|\psi_i|^2\,d\xi\,dz - \frac{1}{e}P(t,1,x)\,u(t,1,x) \\ &+\frac{1}{e}P(t,0,x)\,u(t,0,x). \end{aligned}$$

The two first terms of the right member of (17) are negative: they express the instantaneous dissipation of (16) for any given value of x, whereas the last term expresses the instantaneous exchanged power. Moreover, thanks to the property $\psi(t,0) = 0$, we deduce from (17) the positiveness of the quadratic form $w \mapsto \mathcal{Q}_T(w) := \int_0^T wy\,dt$ for any $T > 0$ and any x: the passive feature of the absorbent wall is restored by model (16).

4. COUPLING WITH AN AEROACOUSTIC WAVE MODEL

As a consequence of (17), the coupling of (16) with another passive dynamic model leads to a globally dissipative system provided that the energy functional is chosen such that the transfer between the two subsystems is balanced. This is highlighted in the sequel.

We consider the aeroacoustic model studied in Mazet [2005]. For simplicity we mainly consider the non convective case with rectangular domain $]0,Z[\times]0,X[$ and boundary control at $z = Z$:

$$\begin{cases} \partial_t\begin{bmatrix} u \\ \rho\end{bmatrix} + c_0\begin{bmatrix} 0 & \nabla \\ \nabla^T & 0\end{bmatrix}\begin{bmatrix} u \\ \rho\end{bmatrix} = f \\ u_x(t,z,0) = u_x(t,z,X) = u_z(t,0,x) = 0 \\ u_z(t,Z,x) = \phi\,y(t,x) \\ w(t,x) = c_0\,\rho_0\,\rho(t,Z,x), \end{cases} \tag{18}$$

where u and ρ are the velocity and the density, f is the perturbation source, c_0 is the velocity of wave fronts inside the fluid medium and ϕ is the porosity coefficient of the porous material. The acoustic energy of $(u,\rho)^T$ is classically given by the functional:

$$E_m = \frac{1}{2}\left\|(u,\rho)^T\right\|^2_{L^2}.$$

We can define a global energy for (16,18) as:

$$E = \int_0^X e(x)E_{\psi_x}dx + \frac{\rho_0}{\phi}\,E_m.$$

We have:

Theorem 6. The system (16,18) is dissipative in the sense:

$$\frac{dE}{dt} = -\int_0^X e(x)\int_0^1\int_0^\infty \xi\,\mu_1\,|\psi_1|^2\,d\xi\,dz\,dx - \int_0^X e(x)\int_0^1\int_0^\infty \xi\,\mu_2\,|\psi_2|^2\,d\xi\,dz\,dx \leqslant 0$$

on any solution $(u,\rho,\psi)^T$ of $(16,18)$.

Proof. From (18) and:

$$\begin{aligned} dE_m/dt &= -\int_0^X\int_0^1 c_0\begin{bmatrix} 0 & \nabla \\ \nabla^T & 0\end{bmatrix}\begin{bmatrix} u \\ \rho\end{bmatrix}\cdot\begin{bmatrix} u \\ \rho\end{bmatrix}dz\,dx \\ &= -c_0\int_0^X\int_0^1(\partial_x\rho u_x + \partial_z\rho u_z + \partial_x u_x\rho + \partial_z u_z\rho)\,dz\,dx \\ &= -c_0(\int_0^1[\rho u_x]^X_{x=0}\,dz + \int_0^X[\rho u_z]^Z_{z=0}\,dx) \\ &= -c_0\int_0^X \rho(t,Z,x)\,u_z(t,Z,x)\,dx. \end{aligned}$$

Note that this global energy dissipation due to wave absorption at the boundary, is not accessible under the frequential or convolution forms (1) and (5). In this sense, the diffusive model (16) is conform to the physical interpretation of such media.

In the convective case (Mazet [2005]), similar results can be obtained up to suitable technical adaptations.

5. DIFFUSIVE REALIZATION OF THE IMPEDANCE OPERATOR

In order to build state realizations of the boundary control defined by (1,3,4), it can be judicious to start from the symbol of the input-output operator $w \mapsto y$, denoted $Q(\partial_t)$ in the sequel. From the numerical point of view, the realization of this operator is more accurate and cheaper than the simulation of the wave propagation inside the medium and so is more adapted for example to control purposes. Let us now compute the symbol of $Q(\partial_t)$.

The equations of the porous medium (1,3) can be written:

$$\begin{cases} \partial_z X = -e(x)\, i\omega\; AX \\ X(0,\omega) = [v, 0]^T, \end{cases} \tag{19}$$

with:

$$X = \begin{bmatrix} \widehat{P} \\ \hat{u} \end{bmatrix},\ A = \begin{bmatrix} 0 & \rho_{\text{eff}}(i\omega) \\ \chi_{\text{eff}}(i\omega) & 0 \end{bmatrix}.$$

Diagonalization of A leads to:

$$A = M \underbrace{\begin{bmatrix} \sqrt{\chi_{\text{eff}}\, \rho_{\text{eff}}} & 0 \\ 0 & -\sqrt{\chi_{\text{eff}}\, \rho_{\text{eff}}} \end{bmatrix}}_{D} M^{-1},$$

where M is the transition matrix; using the change of unknowns $Y = M^{-1}X$, we get the diagonal system of ODEs:

$$\partial_z Y = -e\, i\omega\; DY$$

with limit conditions deduced from (19). After simple computations, we obtain:

$$\begin{cases} \widehat{P}(\omega, z) = \widehat{P}(\omega, 0) \dfrac{A(\omega)/B(\omega, z) + B(\omega, z)}{A(\omega) + 1} \\ \hat{u}(\omega, z) = \widehat{P}(\omega, 0) \sqrt{\dfrac{\chi_{\text{eff}}(i\omega)}{\rho_{\text{eff}}(i\omega)}}\, \dfrac{A(\omega)/B(\omega, z) - B(\omega, z)}{A(\omega) + 1}, \end{cases} \tag{20}$$

where

$$A(\omega) = \mathrm{e}^{2ei\omega\sqrt{\chi_{\text{eff}}(i\omega)\, \rho_{\text{eff}}(i\omega)}},$$
$$B(\omega, z) = \mathrm{e}^{-ei\omega z\sqrt{\chi_{\text{eff}}(i\omega)\, \rho_{\text{eff}}(i\omega)}}.$$

Finally, the input-output correspondence $\widehat{w} \mapsto \widehat{y} = Q(i\omega)\, \widehat{w}$ (which summarizes the frequency behavior of the porous medium) can be deduced by taking $z = 0$ and replacing $i\omega$ by $p \in \mathbb{C}$ in (20):

$$Q(p) = \sqrt{\frac{\chi_{\text{eff}}(p)}{\rho_{\text{eff}}(p)}}\, \tanh\left(e(x)\, p \sqrt{\chi_{\text{eff}}(p)\, \rho_{\text{eff}}(p)}\right).$$

It can be shown that the symbol $\widetilde{Q}(p) = \frac{Q(p)}{p}$ satisfies the hypothesis of theorem 3. From section 2, we then get the following state-representation of $Q(\partial_t)$:

$$\begin{cases} \partial_t \psi = -\gamma\psi + w \\ y = \langle \nu, \partial_t \psi \rangle = -\displaystyle\int \gamma\, \nu\, \psi\, d\xi + \int \nu\, d\xi\, w, \end{cases} \tag{21}$$

where ν is the measure associated to $\widetilde{Q}(\partial_t)$ and γ is a suitable contour.

The presence of complex poles of $\widetilde{Q}(p)$ (see fig. 1) does not allow us to use the same contour γ as in section 3 (note in particular that $\widetilde{Q}(p) \underset{\infty}{\sim} \frac{K_1}{p} \tanh(K_2 p)$). For the realization of $\widetilde{Q}(\partial_t)$, we so consider a contour γ of the form:

$$\gamma(\xi) = |\xi|\; e^{\mathrm{i}\, \mathrm{sign}(\xi)(\frac{\pi}{2}+\alpha)},$$

with $\alpha \in]0, \frac{\pi}{2}]$ (α will be taken equal to $\frac{5\pi}{180}$ for the numerical results of section 6).

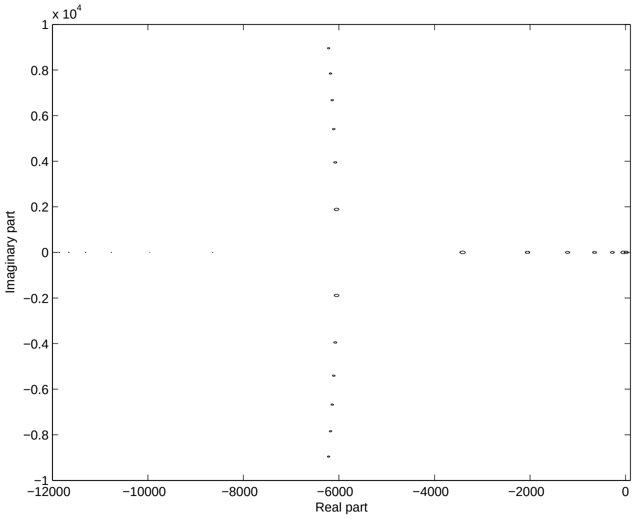

Fig. 1. Poles of $\widetilde{Q}(p)$ ($e = 1$)

6. SOME NUMERICAL RESULTS

As described in section 2, we can perform converging approximations of (16) and (21) by using a discretization $(\xi_l)_{l=1,L}$ of ξ and standard quadratures. We then get approximate dynamic realizations of the form:

$$\begin{cases} \dot{X} = \mathcal{A}\, X + \mathcal{B} w \\ \widetilde{y} = \mathcal{C} X \end{cases} \quad \text{with } X(t) \in \mathbb{R}^L,$$

such that $\widetilde{y} \simeq y$ in a suitable sense (Montseny [2005]).

The frequency responses of the approximations of $H_1(\partial_t)$, $H_2(\partial_t)$ and $\widetilde{Q}(\partial_t)$ respectively obtained with $L = 15$, 20 and 150 are given in figures 2, 3, 4. In the case of $H_1(\partial_t)$ and $H_2(\partial_t)$ (respectively $\widetilde{Q}(\partial_t)$), 4 decades from 10^2 to 10^6 (respectively 3 decades from 10^2 to 10^5) are covered by the ξ-discretization. The numerical values of parameters are:

$\Lambda = \Lambda' = 0.1\, 10^{-3}\,\text{m}$, $\rho_0 = 1.2\,\text{kg.m}^{-3}$, $P_0 = 10^5\,\text{Pa}$
$\mu = 1.8\, 10^{-5}\,\text{kg.m}^{-1}.\text{s}^{-1}$, $\gamma = 1.4$, $\alpha_\infty = 1.3$, $e = 2\, 10^{-2}\,\text{m}$.

Note that by increasing the number of ξ_l, the oscillations of the frequency response of $\widetilde{Q}(\partial_t)$ will be better approximated.

For illustration, the evolution of P obtained from simulation of (16) is given in figure 5. The explicit scheme used for this simulation and whose stability is proved in Casenave [2008], is written:

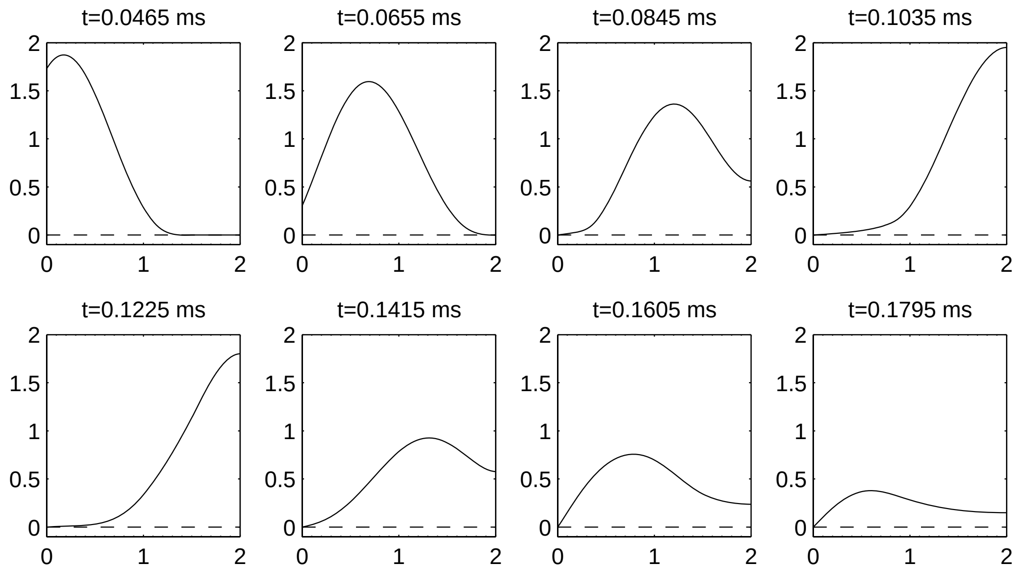

Fig. 5. Evolution of $\widetilde{P} = \sum_l b_{l2} \psi_2(\xi_l)$ (N.B: the unit of length for the z-axis is 10^{-2}m)

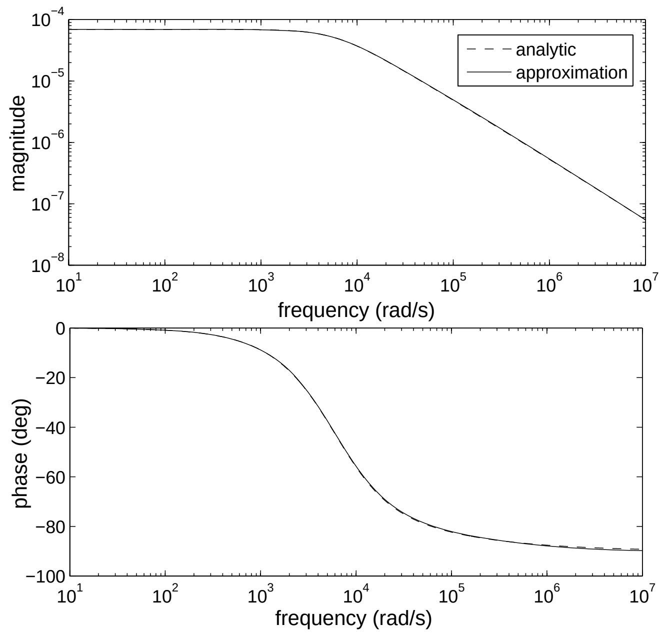

Fig. 2. Frequency response of operator $H_1(\partial_t)$

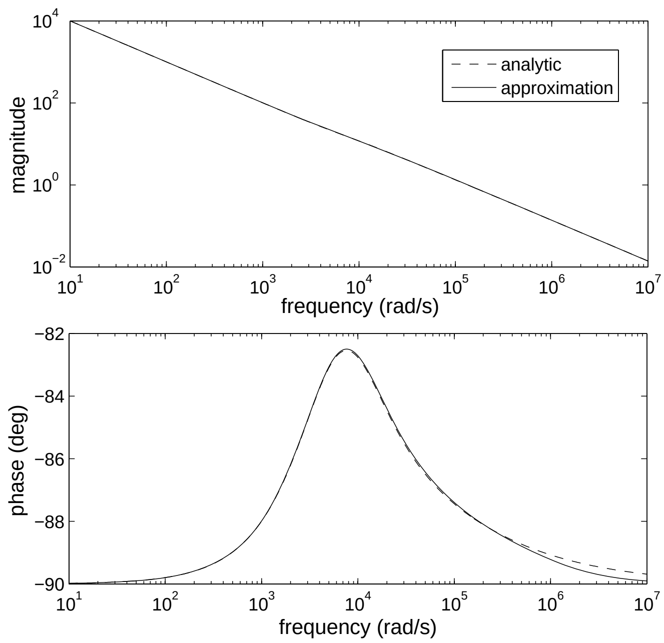

Fig. 3. Frequency response of operator $H_2(\partial_t)$

$$\begin{cases} \psi_1^{n+1}(x,\xi_l) = a_{l1}\, \psi_1^{n-1}(x,\xi_l) - b_{l1}\, G_{21} \sum_j b_{j2}\, \psi_2^n(x,\xi_j) \\ \psi_2^{n+1}(x,\xi_l) = a_{l2}\, \psi_2^{n-1}(x,\xi_l) - b_{l2}\, G_{12} \sum_j b_{j1}\, \psi_1^n(x,\xi_j) \\ \widetilde{u}^{n+1}(x) = \sum_l b_{l1}\, \psi_1^n(x,\xi_l) \\ \widetilde{P}^{n+1}(x) = \sum_l b_{l2}\, \psi_2^n(x,\xi_l), \end{cases} \tag{22}$$

with $\widetilde{u}^{n+1}(x)$, $\widetilde{P}^{n+1}(x) \in \mathbb{R}^K$ and $\psi_1^{n+1}(x,\xi_l)$, $\psi_2^{n+1}(x,\xi_l) \in \mathbb{C}^K$, where $K = 300$ is the number of points of discretization in z, and:

$$a_{li} = e^{-\xi_l 2\Delta_t},\ b_{jk} = \sqrt{c_{jk}\,\frac{e^{-\xi_j 2\Delta_t} - 1}{-\xi_j}},$$

$$G_{21} = G_{12} = \frac{1}{2\Delta z} \begin{bmatrix} 0 & 1 & & & \\ -1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 0 & 1 \\ & & & -1 & 0 \end{bmatrix},$$

where c_{li} are some coefficients computed by simple quadrature of $\int \mu_i(\xi_l)\Lambda_l(\xi)d\xi$, Λ_l being the classical interpolation function:

$$\Lambda_l(\xi) = \frac{\xi - \xi_{l-1}}{\xi_l - \xi_{l-1}} \mathbf{1}_{[\xi_{l-1},\xi_l]}(\xi) + \frac{\xi_{l+1} - \xi}{\xi_{l+1} - \xi_l} \mathbf{1}_{]\xi_l,\xi_{l+1}]}(\xi).$$

The boundary conditions are:

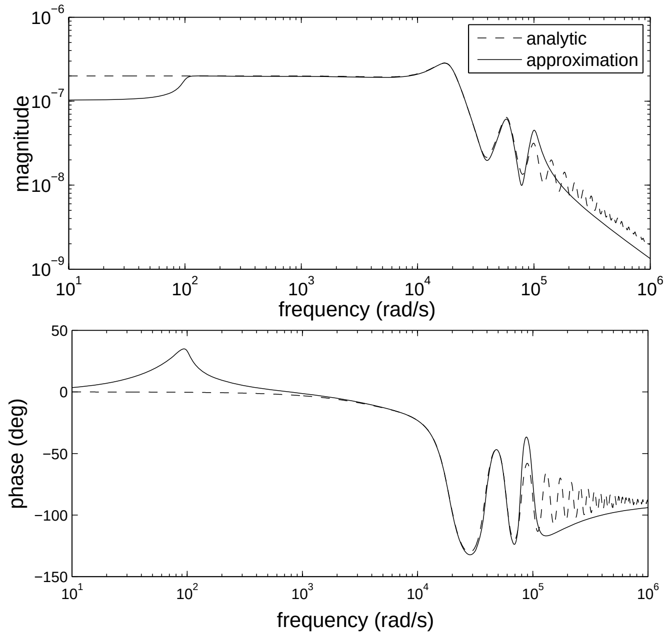

Fig. 4. Frequency response of operator $\widetilde{Q}(\partial_t)$

$$u(t, 1) = 0, \tag{23}$$

$$P(t, 0) = \left(1 - \cos(2\pi \times 1.3\,10^4\, t)\right) \mathbf{1}_{[0,T]}(t) = w. \tag{24}$$

We clearly observe the dissipative and dispersive nature of this propagative model, due to the convolution operators $H_1(\partial_t)$ and $H_2(\partial_t)$.

In figure 6 we can see at a fixed time $t = 0.09\,\mathrm{ms}$, the functions ψ_1 involved in the synthesis of u.

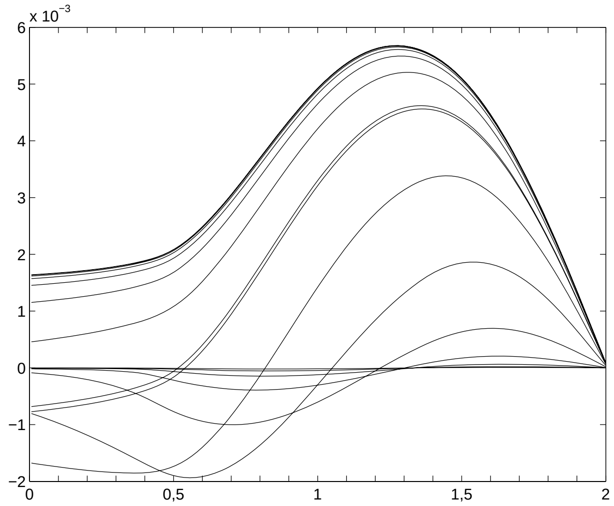

Fig. 6. Functions $\psi_1(t, ., x, \xi_l)$ $l = 1 : L$ at time $t = 0.09\,\mathrm{ms}$

We compare in figure 7 the outputs $y := u_{|z=0}$ obtained by simulation of (16) and (21) with the input w given by (24). Note that the difference between the two results is mainly due to the error approximation of z-discretization. As the simulation of (21) doesn't need any z-discretization, we can consider a high number of ξ_l to approximate the operator Q. Then the simulation of (21) is more accurate but remains cheaper than the scheme (22) (whose global dimension is 10500).

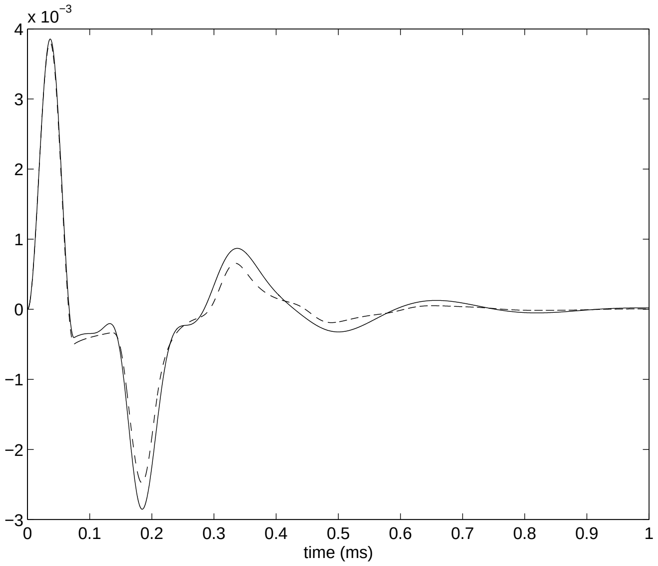

Fig. 7. Outputs y obtained by simulation of (16) (- -) and (21) (—).

7. CONCLUSION

On the basis of the above energy dissipation analysis, we can conclude that the model (1) can be viewed as a dissipative absorbing controller for aeroacoustic waves. Furthermore the quality of the numerical results of the simulation of the boundary operator $w \mapsto y$ confirms that the state-realizations (16) and (21) can be used for more deepened investigations on absorbing boundary control.

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